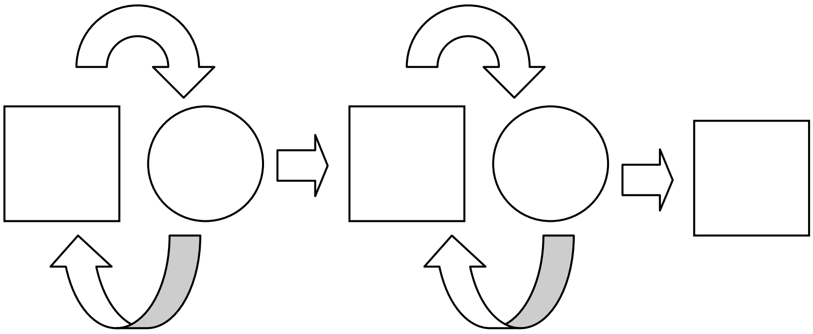
More complex is an interaction scheme with two species: Take the case that we add wolves to the rabbits above and a simple reproduction mechanism that tells us that rabbits split into two by natural fission whilst wolves do so by eating rabbits. This is not *much* more realistic, but we can discuss more complex and interesting forms of interactions now. The appropriate diagram is of the following form then:



Here, the first species displayed stands for rabbit(s), the second for wolves. The upper arrows count as double arrows (producing two samples of a species), while the lower (shaded) arrows indicate one sample starting birth (creation: on the left-hand-side) and one sample starting death (annihilation: one the right hand side). The straight arrows show one-sample-inputs. Hence, the diagram above represents the following three cases of interaction:

(1) *birth*: one rabbit in, two rabbits out (reproduction by fission)

(2) *predation*: one wolf and one rabbit in, two wolves out (reproduction by fission after predation)

(3) *death*: one wolf in, nothing out.

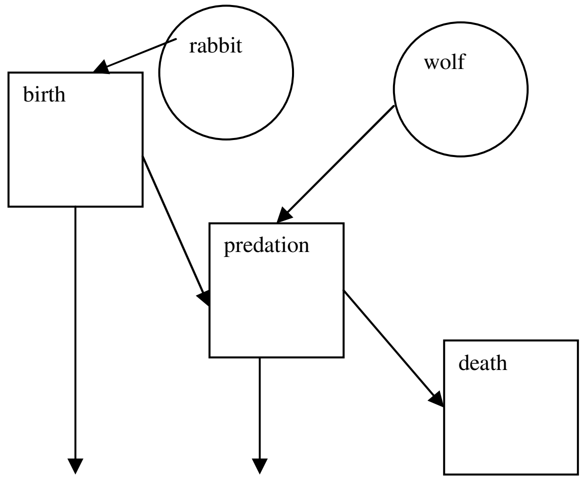
Note that this means that in the simple (still unrealistic) model chosen, rabbits can never die, unless they are eaten by wolves.
These examples give us the chance to actually define Petri nets:

Definition (Petri nets): These consist of a set S of species and a set T of transitions, together with a function i: S x T → N, saying how many copies of each species show up as *input* for each transition, and a function o: S x T → N, saying how many copies show up as *output*, respectively.

However, usually, the transitions take place in a *stochastic* fashion. We have thus a generalized definition:

Definition (stochastic Petri nets): These are Petri nets, together with a function r: T → (0, ∞), giving the *transition rate* (the rate for each transition which can be taken to be a constant).

In other words, we have a *rate equation*, telling us how the *expected number of objects* (copies) of each species changes with time. Even more interesting is the *master equation*, telling us how the *probability that we have a given number of objects* (copies) of each species changes with time. In a sense, the rate equation is deterministic, but approximate, however, if the expected value of the numbers involved is large and the standard deviation is small, then it is quite a good approximation after all. Note also by the way that the process described above can be easily visualized in terms of a graph of Feynman type, in the following way:



The boxes refer (from the left-hand-side to the right-hand-side) to the transitions birth, predation, and death, respectively. The upper inputs are one rabbit (left-hand-side) and one wolf (right-hand-side). Hence, the three line graphs on the left-hand-side refer to rabbits, the three line graphs on the right-hand-side to wolves. The "time coordinate" is running downward. So instead of a Feynman graph for particle interaction, we have now a Feynman graph for rabbit-wolf interactions.

Given three rate constants for the cases discussed here, β, γ, δ, say, we get as rate equations with x(t) being the number of rabbits and y(t) being the number of wolves,

$$dx/dt = \beta x - \gamma xy,$$

$$dy/dt = \gamma xy - \delta y.$$

Here, γ is the explicit *coupling parameter* of the interaction involved. Note that these equations are practically equivalent to the Lotka-Volterra equations, provided the second γ is replaced by some other rate constant ε.[1]

Of course, this can be generalized in a straightforward manner, namely by introducing k quantities of this kind carrying an index in order to denote the type of species: If then $x_i(t)$ is the number of objects of type i at time t, and the transition destroys m_i objects of type i and creates n_i of them, we can write

$$dx_i/dt = r\,(n_i - m_i)\, x_1^{m1} \ldots x_k^{mk},$$

where the somewhat complicated product shows up because a transition occurs at a rate which is proportional to the product of numbers of objects it takes as inputs. The reaction rate is r here. With an appropriate vectorial notation, this can be also written as

[1] In fact, we deal here with a set of two equations: something we call a dynamical system as mentioned earlier. Nevertheless, utilizing the terminological conventions of Baez and Biamonte, we can also talk of one rate equation altogether (thinking in terms of a vectorial expression).

$$dx/dt = r\,(n - m)\,x^m,$$

with the abbreviations: $x = (x_1 \dots x_k)$, $m = (m_1 \dots m_k)$, this is the input vector, and n accordingly, also with x^m in a self-explicatory manner.

As to the master equation, we have to bear in mind that the probability that a given transition occurs within a time Δt is approximately the rate constant for that transition times Δt times the number of ways the transition can occur. Take e.g. the case that we have 10 rabbits and 5 wolves. Then a birth transition can occur in 10 ways, because we have one rabbit as input. Predation can occur in 50 ways, because we have one rabbit and one wolf as input. And death can occur in 5 ways. More generally, the number of ways to choose M distinguishable things from a collection of L is the falling power of the kind

$$L^{\underline{M}} = L\,(L - 1) \dots (L - M + 1).$$

Take a stochastic Petri net now with k species (types) and one transition with rate constant r such that the *i-th* species appears m_i times as input and n_i times as output. We introduce a *labelling* which is a k-tuple of natural numbers $\ell = (\ell_1 \dots \ell_k)$ saying how many things (objects) are in each species. Let $\Psi_\ell(t)$ be the probability that the labelling is ℓ at time t, then the *master equation* can be written as

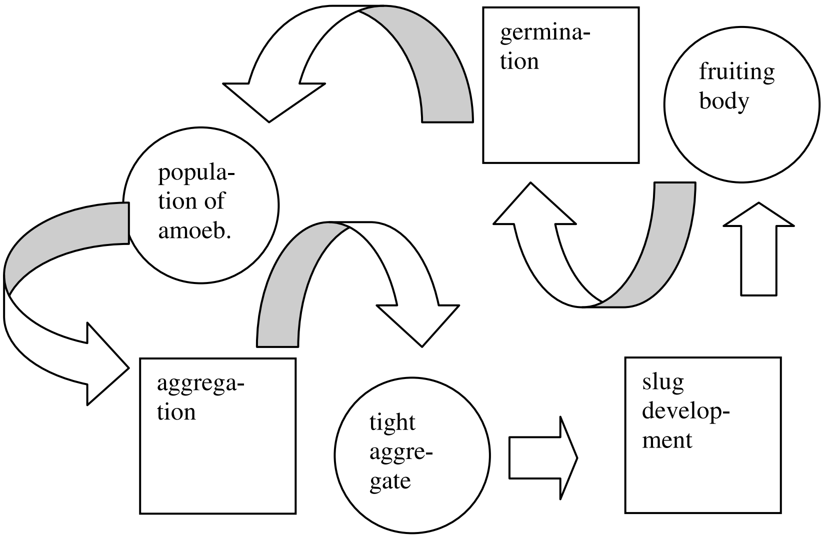
$$d\Psi_{\ell'}(t)/dt = \sum_\ell H_{\ell'\ell}\,\Psi_\ell(t)$$

for some matrix H. This matrix gives all possible transitions among what we can practically call *states* (ℓ', ℓ) of the system in question. We can write this equation without the summation sign Σ, if we utilize Einstein's summation convention: $H^{\ell}{}_{\ell'}\,\Psi_\ell$. When skipping indices, we can also write this in the form

$$d\Psi/dt = H\,\Psi.$$

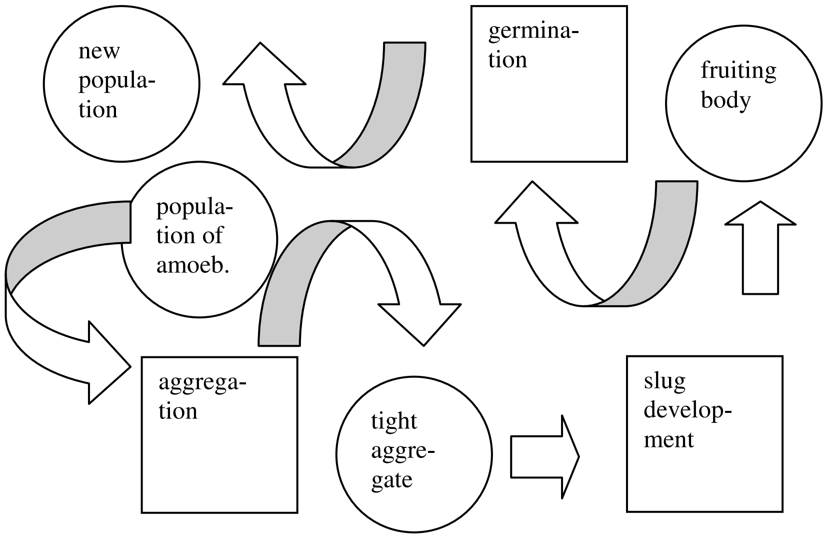
The astonishing point is here that this looks very much like Schrödinger's equation, only that it is defined on a real vector space such that the probability Ψ can be normalized. While the original Schrödinger equation is defined on a complex Hilbert space such that we have to include an imaginary unit on the left-hand-side, and that in this case, the Ψ would be a probability amplitude rather than a probability proper. The probability would be then the absolute value of its square: $P = |\Psi|^2$ – which can be normalized then quite well. The question arises here whether the stochastic model we propose in the above-mentioned can be visualized as a macroscopic approximation of a universally underlying quantum model. Think of the rabbits!

Remembering our slime mould example earlier, we can give a similar description for this more complex system treating amoebae as aggregating agents on the one hand and slime moulds proper as fruiting bodies that produce amoebae again on the other. As to the transitions, it is quite straightforward to visualize the aggregation process as one transition and the germination of spores as another. It might be useful to introduce a third transition for the slime mould itself, differing between the state of a tight aggregate in the beginning and the fruiting body that initializes germination in the end. So in principle, we would expect a layout which looks like the following:

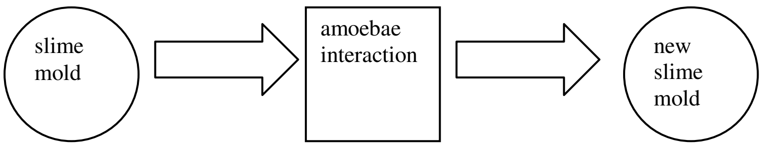


However, the difficulty is here that we do not really know how many amoebae will emerge from germination and how many will be involved in aggregation. Certainly, it must be around ten thousands of samples so that the number of possibilities of how to arrange interactions is considerably increased. On the other hand, large parts of these interactions are actually self-interactions, because it is the population of amoebae that transforms itself by an internal communication that is based on the hormone *chemotaxis*. Hence, visualized as a system for itself, the population of amoebae is in fact self-interacting. Likewise, the aggregate (which is still an agglomeration of individual cells) undergoes a transformation that enables germination at a later stage. Again, the slime mould essentially self-interacts in order to achieve this productive state. In the germination phase, the slime mould ejects a multitude of amoebae (spores) so that this can be visualized as a one-to-many transition. While in the aggregation phase, we have a many-to-one transition. This is indicated in terms of the shaded arrows that represent the input and output for the respective transitions. The developmental transition that enables the slime mould to germinate is indicated by a simple straight arrow in order to stress the point that this is a one-to-one transition.

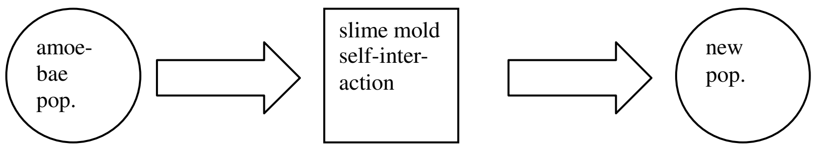
Note two further aspects: First of all, this process is not really a cycle, although it is depicted above in this fashion in order to illustrate the constitutive components involved. But in fact, it is *an open spiral* rather than a cycle, because strictly speaking, the amoebae emerging from the slime mould fructuation process are not the same that aggregate to built up the slime mould in the first place. So what we have is more the following picture which is more precise as to the description of the dynamics:



But then, we recognize that the scope of the processes involved can be represented in a simplified way, according to whether we choose the perspective of the *micro-level* or *macro-level*, respectively: If we take the perspective of the slime mould, then one part of the (spiral) cycle can be visualized as a self-interaction that reproduces slime moulds:



But if we take the perspective of the amoebae, then we can visualize the same process as a self-interaction of the population that reproduces amoebae:



While the main action of the first diagram is "germination", the main action of the second diagram is "aggregation". However, the latter is based on an internal interaction within the population of amoebae. Hence, it can be interpreted as *communication*. (This is in fact what happens: Chemotaxis is a procedure of communication in the first place.) Note that Petri nets tell essentially how one object performs a transition (by suitable interactions) and becomes another object. A communicating amoeba is still an amoeba, but its action is altered. It is in this sense that we can say that an amoeba undergoing chemotaxis is *another* amoeba.[2]
But remember that a slime mould is an *organism* in the first place, but also a collective of individual amoebae at the same time. If the tight aggregate transforms itself by means of chemical processes, then this can also be visualized as interaction (i.e. communication) of amoebae, eventually resulting in the fruiting body which is able in turn to proceed to germination.

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The advantage of what Baez and Biamonte offer us is that we can utilize the methods from the field of quantum field theory in order to understand the stochastic processes better: The first, very useful probability distribution that can be discussed in terms of Petri nets is the *Poisson distribution* well-known from many fields in physics and elsewhere. Baez and Biamonte discuss this in terms of an example which is dealing with the capture of fish (very funny: French for "fish" is "poisson"[3]).
The idea is the following: The probability for one fish caught within time Δt is $r\, \Delta t$. The probability for n fish being caught is $\Psi(n, t)$ accordingly. All such probabilities can be summarized in terms of a power series of the form

$$\Psi(t) = \sum_0^\infty \Psi(n, t)\, z^n,$$

where we call z the *generating function*. Now, recall that the master equation can be written in a form which is similar to Schrödinger's equation: $d\Psi/dt = H\, \Psi$. Traditionally, we call H the *Hamilton operator* (or: Hamiltonian). In our case here, this equations describes how the probability of having caught any given number of fish changes with time. However, in quantum physics we discuss the creation and annihilation of particles. Thinking of fish instead, we can express the fact that we can be sure at time t to have n fish by writing

$$\Psi = z^n.$$

Creation of particles is given by the creation operator of the form $a^\dagger\, \Psi = z\, \Psi$. Hence, one more fish is consequently expressed by

$$a^\dagger\, \Psi = z^{n+1}.$$

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[2] In fact, this is an important point, once we come to social systems: because a person remains a person, but its significant properties change according to the interaction a person is undergoing. And the main type of interaction for human beings is communication. But if the significant properties change, we can plausibly argue that we deal with another person. Probably, this becomes more plausible when we speak of a person's change of state that alters the person and thus produces a new one.

[3] The distribution is actually called after the mathematician Siméon Denis Poisson who lived from 1781 until 1840.

And the probability of having caught n fish by time t is given by the distribution

$$[(rt)^n/n!] \exp(-rt),$$

which is called *Poisson distribution*. We find that this result is compatible with choosing the Hamiltonian such that $H = r(a^\dagger - 1)$. This also solves the master equation whose general solution is $\Psi(t) = \exp(t H) \Psi(0)$ with $\Psi(0) = 1$. Remember that the Hamiltonian for macroscopic everyday problems is actually a matrix.
Comparing then stochastic (macroscopic) dynamics with quantum dynamics, we realize that in probability theory, the passage of time is described by a map that sends probability distributions to probability distributions. This can be described by a *stochastic operator* of the form

$$U: L^1(X) \to L^1(X)$$

which is linear such that $\int U \Psi = \int \Psi$ and $\Psi \geq 0$, $U \Psi \geq 0$. While in quantum physics the passage of time is described by a map that sends wave-functions to wave-functions, which can be expressed in terms of an *isometry*

$$U: L^2(X) \to L^2(X)$$

that is also linear in the sense that $\langle U \Psi, U \Phi \rangle = \langle \Psi, \Phi \rangle$. If these isometries have inverses, they are called *unitary* operators. (Time evolution in quantum physics is usually reversible. In probability theory it is usually not.)[4]
In quantum physics, the solution of the Schrödinger equation is mainly expressed by the term $\exp(-i t H)$, and a Hamilton operator that makes this term unitary for all t is one which is *self-adjoint*: $\langle H \Psi, \Phi \rangle = \langle \Psi, H \Phi \rangle$. So what properties should a Hamilton operator possess in order to make $\exp(t H)$ stochastic?
Now what we find is that we must have

$$\int \exp(t H) \Psi = \int \Psi.$$

We can also recognize that the condition $\Psi \geq 0 \Rightarrow \exp(t H) \Psi \geq 0$ is satisfied, if we introduce the concept of an *infinitesimally stochastic matrix* H: This is one whose columns sum to zero and whose off-diagonal elements are non-negative.
Let us come back now to our rabbits: In a given population of rabbits, we call Ψ_n the probability of having n rabbits. We utilize then the power series expansion of the form

$$\Psi = \sum_0^\infty \Psi_n z^n.$$

So what we do here is essentially to sum over all possible probabilities. The advantage is in the fact that we can define creation and annihilation operators on formal power series such that

$$a \Psi = d/dz \, \Psi,$$

$$a^\dagger \Psi = z \Psi,$$

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[4] We use here Dirac's bracket notation and come back to that later. The full bracket is a scalar product.

where the first expression gives the *annihilation operator* and the second the *creation operator*. We have already seen above how creation works by producing one new sample, e.g. one new rabbit. The annihilation procedure is a little more complicated, because we have to think of the n ways we could pick a rabbit and make it disappear. We have thus:

$$a\,\Psi = n\,z^{n-1}.$$

Note that we have also

$$(aa^\dagger - a^\dagger a)\,\Psi = d/dz\,(z\,\Psi) - z\,d/dz\,\Psi = \Psi,$$

hence, creation and annihilation operators do not commute: $[a, a^\dagger] = 1$. This essentially means that there is one more way to put a rabbit in a cage of rabbits, and then take one out, than to take one out and then put one in. Clearly, the evolution of the probabilities summarized in Ψ follows the rate of change of Ψ according to $d/dt\,\Psi = H\,\Psi$. The details depend on the situation chosen. Baez and Biamonte give a number of possible scenarios utilizing the simplified rabbit system: So catching a rabbit is described by the Hamiltonian $H = a^\dagger - 1$, a dying rabbit by $H = a - N$, where N is the *number operator*: $a^\dagger a$, breeding a rabbit by $H = a^\dagger a^\dagger a - N$, and so forth. The general rule is as follows: Suppose we have a process taking k rabbits as input and having j rabbits as output, then the respective Hamiltonian possesses the form $H = a^{\dagger j} a^k - N\,(N - 1)\,\dots\,(N - k + 1)$.

*

Recall from above that for our system of rabbits and wolves, we can write down the rate equations in terms of given rate constants as

$$dx_1/dt = \beta\,x_1 - \gamma\,x_1\,x_2,$$

$$dx_2/dt = \gamma\,x_1\,x_2 - \delta\,x_2.$$

The Hamiltonian for the master equation becomes more complicated, but without giving further details here[5], we can write it down for the model discussed here in the form

$$H = \beta\,B + \gamma\,C + \delta\,D,$$

where $B = a_1^{\dagger 2}\,a_1 - a_1^\dagger\,a_1$, $C = a_2^{\dagger 2}\,a_1\,a_2 - a_1^\dagger\,a_2^\dagger\,a_1\,a_2$, $D = a_2 - a_2^\dagger\,a_2$. In other words: Birth annihilates one rabbit and creates two rabbits. Predation annihilates one rabbit and one wolf and creates two wolves. Death annihilates one wolf. If we solve now the master equation accordingly such that we have $\Psi(t) = e^{tH}\,\Psi(0)$, then we can utilize the fact that

$$e^{tH} = 1 + t\,H + (t\,H)^2/2! + \dots$$

to multiply this with $\Psi(0)$ to get $\Psi(t)$ altogether. All the possible products of B, C and D involved can be drawn as Feynman diagrams, or to be more precise: as a sum of Feynman diagrams (as shown earlier).

Now, the interesting point is that the type of stochastic mechanics (or dynamics rather) we have discussed so far admits an *analogue of Noether's theorem*. In particular, this is true for Markov processes in general, of which stochastic Petri nets turn out to be a special case. If we

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[5] See Baez, Biamonte, op. cit., 74-76.

consider a set of states X, then a Markov process is described by a real matrix $H = (H_{ij})$, $i,j \in X$. If we assume that the system is in state i, then the probability of being in some state j after some time changes with time, and the H_{ij} is defined to be the time derivative of this probability. From here, we can easily motivate again the introduction of "infinitesimal stochasticity": Given a finite set X, a matrix of real numbers H is *infinitesimally stochastic*, if $i \neq j \Rightarrow H_{ij} \geq 0$, and $\sum_i H_{ij} = 0$ for all $j \in X$. A Noether theorem applied to Markov processes tells us now that an observable commutes with the Hamiltonian if and only if (iff) the expected values of that observable and its square do not change with time. Or in other words, if O is the observable,

$$[O, H] = 0 \text{ iff } d/dt \int O\, \Psi = 0 \textit{ and } d/dt \int O^2\, \Psi = 0,$$

for all Ψ that satisfy the master equation. (In a sense, it looks remarkable that we have to take care of the square of O here, not only of O itself. But this is due to the difference we have already noticed when comparing stochastic mechanics with quantum mechanics. We leave out the proof here and instead refer back to our primary source.)
In order to directly compare the Noether theorem versions in question here, we state the quantum version and the stochastic version one after the other:

Theorem A: Let X be a finite set. Suppose H is a self-adjoint operator on $L^2(X)$, and let O be an observable. Then $[O, H] = 0$ iff for all states Ψ satisfying Schrödinger's equation such that $d/dt\, \Psi = -\, i\, H\, \Psi$, the expected value of O in state Ψ does not change with time t.

Theorem B: Let X be a finite set. Suppose H is an infinitesimally stochastic operator on $L^1(X)$, and let O be an observable. Then $[O, H] = 0$ iff for all states Ψ satisfying the master equation such that $d/dt\, \Psi = H\, \Psi$, the expected values of O and O^2 in the state Ψ do not change with time t.

*

Hence, in principle, what we do here is to compare approaches that rely on self-adjoint operators (in the case of quantum mechanics) and on infinitesimally stochastic operators (in the case of stochastic mechanics), respectively. This turns out to be very important for network theory, because there is a class of operators that combines both properties: Such operators are called *Dirichlet operators*. Hence, the operator H is said to be *self-adjoint*, if it equals the conjugate of its transpose: $H_{ij} = \overline{H}_{ji}$. And the operator H is said to be *infinitesimally stochastic,* if its columns sum to zero and its off-diagonal elements are non-negative. So H is a Dirichlet operator, if it is both self-adjoint and infinitesimally stochastic. We can formulate the following then:

Theorem C: Any finite simple graph with edges labelled by positive numbers gives a Dirichlet operator, and conversely.

Suitable networks of this kind can be described in terms of circuits that consist of resistors whose conductance is given by the labelling.[6] But it is also interesting to apply the insight gained to reaction networks of various kinds, particularly of chemical type.
We take a simple reaction as an example:

$$A \rightarrow B,$$

[6] Cf. Baez, Biamonte, op. cit., 143 sqq.

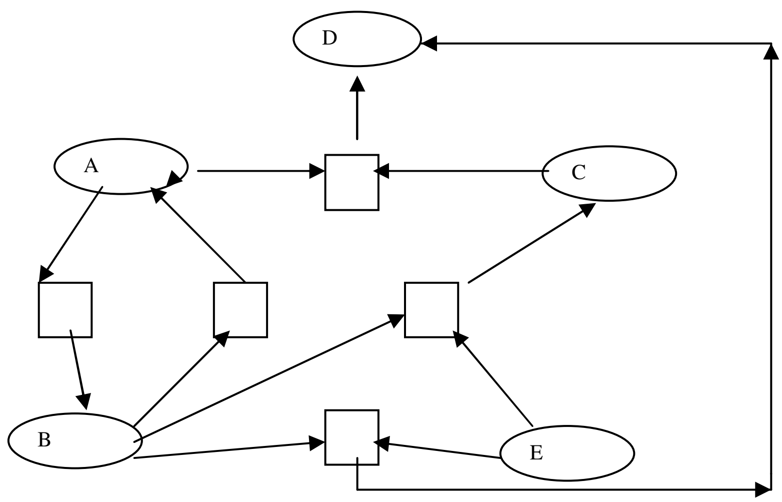
$$B \rightarrow A,$$

$$A + C \rightarrow D,$$

$$B + E \rightarrow A + C,$$

$$B + E \rightarrow D.$$

Equivalently, this can be visualized in terms of a Petri net of the following kind:



We have replaced here lines and arrows for the block arrows used earlier. Obviously, each reaction corresponds to a transition of this Petri net. We recognize the complexes here in which each species shows up several times. Hence, essentially, *a reaction network is a graph whose vertices are labelled by complexes*. On the other hand, a reaction network can also be visualized as a set of species together with a directed multi-graph whose vertices are labelled by complexes of those species. In this sense, it is also the generator of a Petri net and vice-versa. And if each reaction is labelled by a rate constant, the reaction network is said to be stochastic. (We can realize here the relevance of category theory again, because we can define suitable morphisms that map one type of network onto the other.)
Let us now define the *deficiency* of a network: This is the number of vertices minus the number of connected components minus the dimension of the stoichiometric subspace. Note that two vertices lie in the same connected component, iff you can get from one to the other by a path that is direction-independent. (In our example, there are five vertices and two connected components, depicted in the following fashion:

$$A \Leftrightarrow B$$

$$\begin{array}{ccc} A + C & \Rightarrow & D \\ \Uparrow & & \Uparrow \\ B + E & \Leftrightarrow^{id} & B + E \end{array}$$

The lower arrow is the identity here.)

The stoichiometric subspace of a reaction network is the subspace Stoch $\subseteq R^S$ spanned by vectors of the form x – y where x and y are complexes connected by a reaction. In our example, each complex can be seen as a vector in R^S which is a space whose basis can be visualized as A … E. So each reaction gives a difference of two vectors with respect to the complexes: The top reactions give B – A and A – B, respectively. The central part gives D – A – C. The lower part gives on the left-hand-side A + C – B – E, while the right-hand-side gives D – B – E. These five vectors span the stoichiometric subspace. But because these vectors are linearly dependent, the subspace is three-dimensional rather than five-dimensional. Hence, in a sense, *the stoichiometric subspace is the space of ways to move in R^S that are allowed by the reactions in the given network*. So in the end, we find that the deficiency of our network example is 5 – 2 – 3 = 0.
Now then, we call a network *weakly reversible*, if whenever there is a reaction going from a complex x to a complex y, there is also a path going back from y to x. Hence, our network example is not weakly reversible, because we can go from A + C to D, but not back (and so forth). So we formulate the

Theorem D: Given a network with a finite set of species S. Suppose its deficiency is zero. Then:

(1) If the network is not weakly reversible and the rate constants are positive, the rate equation does not have a positive equilibrium solution.

(2) If the network is not weakly reversible and the rate constants are positive, the rate equation does not have a positive periodic solution.

(3) If the network is weakly reversible and the rate constants are positive, the rate equation has exactly one equilibrium solution in each positive stoichiometric compatability class. This equilibrium is complex balanced. Any sufficiently nearby solution that starts in the same stoichiometric compatability class will approach this equilibrium as t goes to infinity. There are no other positive periodic solutions.

In other words: The interesting dynamics happens in networks that have not deficiency zero. The first condition of part (3) is a consequence of the fact that if Stoch $\subseteq R^S$ is a stoichiometric subspace, and $x(t) \in R^S$ is a solution of the rate equation, then x(t) always stays within the set x(0) + Stoch. This is called the *stoichiometric compatability class* of x(0). While the complex balance entails that we can turn the equilibrium solutions of the rate equation into those of the master equation. If we would prefer to have a compact version of what we have done so far, we could introduce a very compact diagram that summarizes the information in a stochastic reaction. Take the map $Y: K \to N^S$ sending each complex to the linear combination of species that it is composed of. Then the required diagram is of the form

$$(0, \infty) \leftarrow^{r} T \Rightarrow_{t}^{s} K \rightarrow^{Y} N^{S}.$$

We have utilized here the definition of a reaction network in a more formal fashion, namely as a triple (S, s, t: T → K) such that S is a finite set of species, T a finite set of transitions, and K a finite set of complexes, together with source and target maps s and t. In particular, each transition τ gives a vector

$$\partial\tau = t(\tau) - s(\tau) \in R^K$$

that tells us what change in complexes it actually causes. In fact, ∂ can be extended (as all the other maps) to a linear map so that we can follow the mathematicians and call it *boundary operator*. Note that a reaction network has deficiency zero, iff $Y(\partial\rho) = 0 \Rightarrow \partial\rho = 0$ for every $\rho \in R^T$. (And it actually follows that the deficiency of a reaction network is the dimension of $\mathrm{im}\partial \cap \ker Y$. Indeed, $\mathrm{im}Y\partial$ is nothing but the stoichiometric subspace mentioned above.) We can compute the deficiency then by the number of vertices in the network minus the number of connected components minus the dimension of $\mathrm{im}Y\partial$. We know that for our last example, this is just zero. We can then also give the

Theorem E: A weakly reversible network with zero deficiency given. Then for any choice of rate constants there is an equilibrium solution of the rate equation where all species are present in nonzero amounts.

Here, the important (and sufficiently innovative) aspect is that the rate equation for a reaction network looks like

$$dx/dt = Y\,H\,x^Y,$$

where Y is a matrix now such that the equation becomes non-linear! The equilibrium would be given by $dx/dt = 0$ so that we should look for a solution of $H\,x^Y = 0$. This is mainly achieved by finding all solutions of $H\,\Psi = 0$ first, and then also those for which $\Psi = x^Y$. The relevant information for doing so is contained in the sequence shown above so that we get the finite sets of transitions (T), complexes (K), species (S) plus the rate constant for each transition given by r, the source and target maps s, t, as well as the injection Y which tells us how each complex is made of species. Utilizing some knowledge from the handling of sequences (of which we will leave out the details here[7]), we can actually reproduce the desired equation of the indicated type. By replacing addition by multiplication and multiplication by exponentiation, we also achieve a generalized type of matrix operations such that we can write:

$$x^Y = (x_1, \dots x_k)^{\begin{pmatrix} Y_{11} & \dots & Y_{1\ell} \\ \dots & & \dots \\ Y_{k1} & \dots & Y_{k\ell} \end{pmatrix}}$$

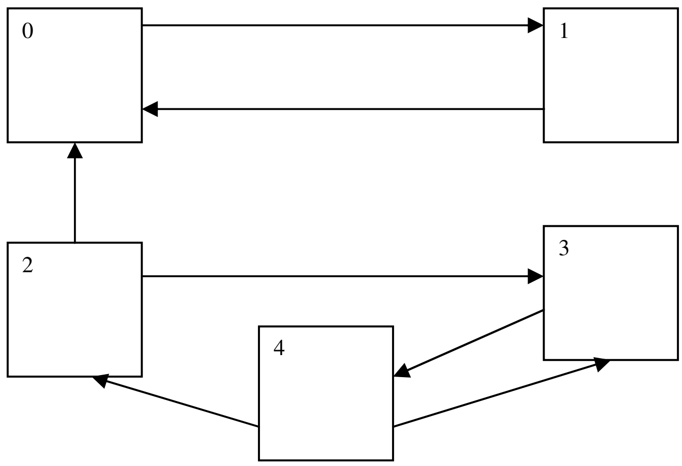
so that $x^Y = (x_1^{Y11} \dots x_k^{Yk1}, \dots, x_1^{Y1\ell} \dots x_k^{Yk\ell})$. The entries of the matrix Y tell us how many times each species shows up in each complex. Or in general: If you have a certain number of things of each species, then we can list these numbers such that the matrix formed describes in how many ways one can built up each complex from the available things. We can still show the equivalence of three expressions:

$$dx/dt = Y \sum_{\tau\in T} r(\tau)\,(t(\tau) - s(\tau))\,x^{Ys(\tau)} = Y\,(t - s)\,s^{\dagger}\,x^Y = Y\,H^Y.$$

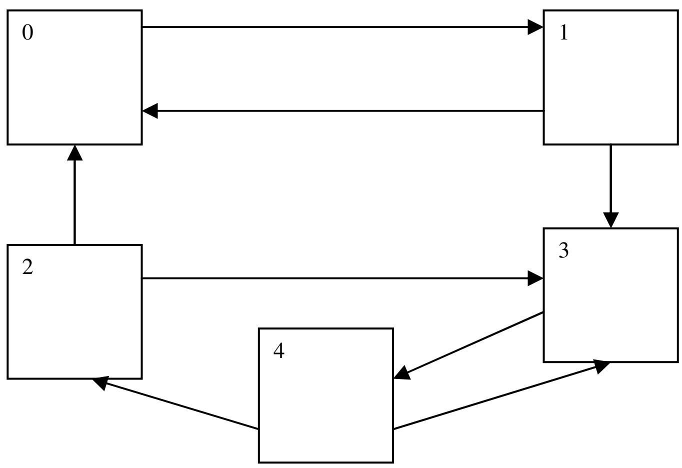
It is important to remark that these relationships fall into a field that is closely related to category theory. In fact, the mappings involved can be visualized as arising from a pair of adjoint functors.
Now, if we introduce graphs that are weakly reversible, i.e. such that for every edge $\tau: i \rightarrow j$, there is a directed path going back from j to i, meaning that we have edges $\tau_1: j \rightarrow j_1, \dots \tau_n: j_{n-1} \rightarrow i$. The advantage of the weakly reversible case is that we get one equilibrium solution of the master equation for each component of our graph, and all equilibrium solutions are

[7] But see Baez and Biamonte, op. cit., 207-209.

linear combinations of these. Note that the following graph (where we use boxes for the acting complexes as an exception) is *not* weakly reversible while the second actually *is*:



We can label the edges with rate constants (from above: 1, 1, ½, ½, 1, ½, ½ , say). The second diagram is:



Here, the labels of the second edge from above and of the new one on the right-hand-side are ½ each.
Utilizing this insight, we can re-phrase earlier results in the following way:

<u>*Theorem F*</u>: Let H be the Hamiltonian of a weakly reversible graph with rates

$$(0, \infty) \xleftarrow{r} T \underset{t}{\overset{s}{\rightrightarrows}} K.$$

Then for each connected component $C \subseteq K$, there is a unique probability distribution $\Psi_C \in R^K$ that is positive on that component, zero elsewhere, and is an equilibrium solution of the master equation $H\Psi_C = 0$. Moreover, such distributions form a basis for the space of equilibrium solutions of the master equation. So, the dimension of this space is the number of components of K.

We also note from the above that we can formulate

Theorem G: The Hamiltonian for a graph with rates is given by $H = \partial s^\dagger$.

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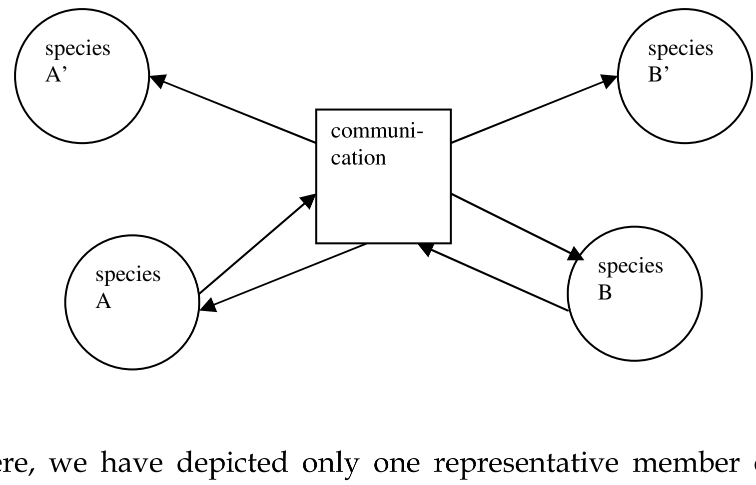
While within the framework of the course offered by Baez and Biamonte, many further details, and in particular: the proofs of the theorems we have only mentioned here, are displayed, we would like to end with a review of possible applications of all of that to urban social systems. In that case, we do not deal with chemical compounds or similar stuff, but with a large number of systemic aspects that remain more or less invariant. Essentially, a city (or one of its living quarters as to that) consist(s) of inhabitants (i.e. individual persons) that can be grouped as species according to their membership in pre-defined social groups. We have networks on two different levels: First of all, there is a network whose vertices (if presented in terms of a graph) are persons (= agents operating on other agents) and whose edges are the interactions among them. But there is also a network on a higher level whose vertices are the groups and whose edges are the (average) interactions among these groups. Obviously, within this picture, a society is a group of groups. If we call the persons *objects* and their interactions *morphisms*, then we can define a *category* for a group (for one species). If the objects are groups (categories), then the morphisms are *functors* between (various) categories. Interactions among persons, i.e. among human beings, are mainly governed by language, i.e. by linguistic operations. Usually, if p is a person and q another, then we say that p and q interact, if they talk to each other or communicate otherwise, but in a linguistic manner. Note however that different from formal languages that possess a lexicology and a syntax, in everyday languages there is also a semantic such that very often, the interaction $p \rightarrow q$ (p talks to q) is not equivalent to the interaction $q \rightarrow p$ (q talks to p).
If visualized in terms of Petri nets, we can say the following: that social interactions of this type can be understood as transitions such that one member of a species that interacts with a member of another species *changes in time* accordingly. In other words, we can apply the terminology of creation and annihilation operators, provided we interpret a person that has interacted as a *new* person. If p and q converse, then *after* their talk, p and q have been transformed, namely by means of the information flow which is represented by this process of communication. Usually, in social systems, communication is nothing but transport of information. As seen under the perspective of one specific group (species), information transport is percolation within the network. Hence, given the case that several members of different groups communicate with one another, we have the following situation:



Here, we have depicted only one representative member of the respective group, but in principle, the arrows can be multi-valued. Communications acts from group A to group B and vice-versa. While interacting, both groups change, because the communicating members of them change. We can formally say that new persons are actually produced by means of the ongoing discourse. But different from the chemical or biological examples given earlier, in this case, production means producing a new inventory of behaviour and thus language! In a social system this is because linguistic (and behavioural) context determine cognition and (further) communication (therefore also cooperation in the long run). It is because human beings have no choice: *they must communicate* all the time, and thus they change all the time. (Sartre!) Hence, species A *after communication* is a new species A′, and the same is true for species B which changes into B′. However, note also that the group (species) itself resembles very much the organism we encountered in terms of the process of slime mould aggregation: This is because the group practically constitutes the organism. In other words: Persons communicate and, by doing so, aggregate (accumulate) to form a group which is representing their collected interests and opinions. These latter can be read from the explicit *discursive structure* the group is utilizing. And this structure is expressed by characteristics that can be obtained from linguistic aspects. But at the same time, the discursive structure is also mapped on the explicit modes of behaviour such that spatial design e.g. can also be visualized as an outcome of this very discourse.

So what we have is a system with two organizing levels: On the group level, we have a spirally organized process of aggregation which can be viewed as a type of self-interaction reproducing the group all the time, and, by doing so, reproducing the individual members. This is a strict parallel to what we have said earlier with respect to the two perspectives under which slime mould aggregation can be visualized in terms of a reproductive procedure. In fact, in the case of human beings, this parallel can be taken in a literal sense, as far as biological reproduction is being concerned[8]. But we talk about *social* reproduction here in much more general terms.

Hence, if we talk about social groups very much in the manner of a finite set of interacting species, then we would like to apply theorem D (3) in order to find out whether there are states of the system that provide an appropriate equilibrium solution for the master equation involved.[9] This can be done in the following way:

First of all, we have to stratify somewhat the group structure of urban social space. For simplicity, we leave out some possible complications. Hence, we classify the groups according to their referring to characteristics that are related to kinship and descent (species type A), occupation (type B), further stratified according to whether one works in a private sector (type B1) or in a public service (type B2), education (type C), cultural aspects (type D), and (other) interests in an unspecified sense (type E). Note that we neglect age and gender as group characteristics. We also do not talk on hierarchies which are especially immanent in various species in different ways. The most obvious case is the formalized hierarchy and ranking of positions in the (professional) occupational field. The idea is then that these groups are visualized as species in the sense of what we have discussed here. Obviously, all these groups are constituted by individual persons who interact within the given conditions of their respective networks. These persons are permanently engaged in communication which determines their transitions in detail according to mainly linguistic criteria. On the

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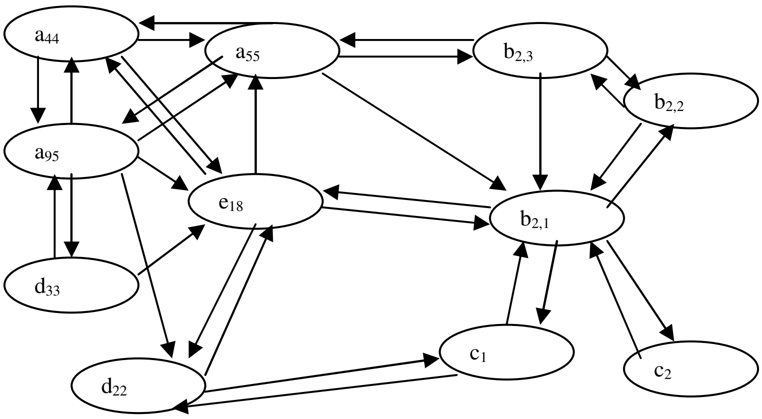
[8] Indeed, an individual human being can be visualized as a collective of several types of cell populations in turn consisting of individual cells that are cooperating to produce and then maintain an organism. Hence, biological reproduction of humans can be viewed under a perspective of some special aggregation process, except that the concrete production phase is usually limited to a pairing of exactly two cells only.

[9] Note that an equilibrium solution of the master equation does not entail a static or stationary situation with respect to the evolving system: It simply means that there is a stationary value for the probability of a specific state of that system.

other hand, on the next level of complexity, groups can communicate among each other and cooperate e.g. for given purposes. In that case, we have a situation which is similar to the complexes discussed earlier: It is important to notice that in a group coalition (or complex) merely communicative processes are transformed into *political* processes. The difference is in the transcending quality of interests pursued, because the groups involved in such a complex have established a consensus about carrying their intentions into public space rather than concentrating their activities to within the limits of their specific social sector.

Take an average urban quarter in a large metropolitan area. Typically, a *living quarter* is the administrative unit on which the urban organization can be easily based.[10] The quarter itself is based on the smallest unit which is a *household*. In fact, for a city like Berlin, a typical metropolitan district, Schöneberg say, is of an area size of roughly ten square kilometres with approximately 115.000 inhabitants.[11] We can differentiate here among ten living quarters one of which is what we call "the island". This is essentially an area in the shape of a triangle that is cut out by the railway trail of three different metropolitan lines. The island itself has about 13.000 inhabitants.

If we assume that group types A through E show up in terms of numbers of species such that there are *a* groups of type A, b_1 and b_2 groups of type B and so forth up to *e* groups of type E, then a large part of them is probably involved in the establishment of coalitions to form political complexes. Utilizing the diagram terminology we have introduced earlier, we might be able to express such a situation in the following way:



Here we have assumed that a sufficiently realistic picture would be provided by setting a ≈ 100, $b_1 \approx 30$, $b_2 \approx 10$, c ≈ 5, d ≈ 50, e ≈ 30. In other words, we deal with 225 interacting species altogether. But we have to note that most persons belong to more than one species. On the average we have 50-60 members per species. The social networking of individual group members we will call *communication* proper, while the interaction of groups we will call *policy* instead. For the diagram we have selected 11 such groups of various types.[12] The lines and arrows among them display the interactions such that each line is thought of as representing an appropriate transition normally depicted by a labelled box which we have

[10] Officially, Berlin comprises 97 living quarters altogether.

[11] We talk here about the original city structure of Berlin in the sense of the first composition of twenty administrative districts in the year 1920. The number of these units has been reduced to 12 more recently.

[12] Here, the groups numbered with utilizing a comma shall be differed from those numbered by a two-digit number.

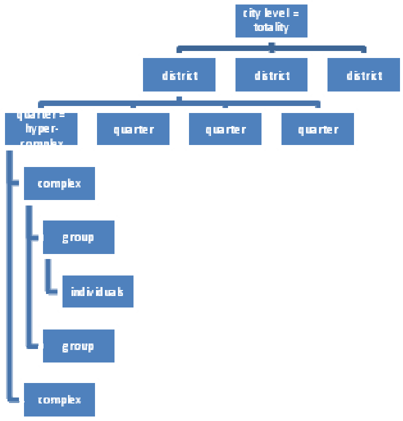
left out here for simplicity. The idea is that for this example (i.e. for these 11 out of 225 groups) a complex is forming following a policy that is put forward by the groups based on their specific group consensus. Other complexes in other group combinations form similarly so that the community of the living quarter in question can be visualized as a *hyper-complex* which is forming out of the complexes involved. So on the first level of communication, individual persons form groups, on the second level, groups form coalitions (complexes), and on the third level, complexes form hyper-complexes. As to their inventory of behaviour and discourse, the latter are the concrete expression of the community that can be observed (in political terms) within a living quarter.[13] As to the diagram displayed above, it is necessary to remember that what was called "stoichiometric subspace" is related here to a space of free play that provides the available range of possible motions within the given framework of state probabilities. Each group possesses at least one incoming and one outgoing arrow. Hence, it is secured that for each path there is a return-path (not always directly) such that we can establish a condition which is equivalent with the condition of weak reversibility. In other words, the complex that forms according to the diagram shown is on the best way to achieve a desired equilibrium. This can be visualized as the condition for a transitory type of structural stability for a complex in the first place. So what we would expect is a similar structure for all the other complexes present.[14]

This implies a *strategic* aspect of the policies involved: Enduring transitory stability can be achieved, if the social design is such that the field of possible transitions remains well-balanced. (If one or the other connection is vanishing, it must be clear that others have to be established in order to stabilize the network.) Of course, we have neglected here the interaction with the other complexes which make this special complex (like all the others) an open system sensible against flow of energy and flow of information across the system boundary.

If we assume that 11 species form a complex as shown above, then about 20-30 other such complexes can possibly form in order to compose a hyper-complex. (For a living quarter, this would be the level of groups of groups. Of course, for a whole city we would have another level of grouping living quarters to form a district, and a level of grouping districts to form the (administrative) totality of the city. So for a usual city, we would have at least five levels of organization above the individual inhabitants in the following fashion:

[13] For the example chosen, size and number of inhabitants reflect well the conditions of ancient Athens by the way. Hence, it is very tempting to look for explicit applications of designing the interior tensions of this social space by positioning the poles of *agorá* and *theátron* as explained by Richard Sennett. I have elaborated on that in more detail in: Η ΝΕΑ ΠΟΛΥ, op. cit. (LIT, Berlin, 2014)

[14] It is unlikely however that the deficiency of the graph would be zero, despite the many vectors that are linearly dependent. But the dim(im $Y\partial$) is comparatively large. What we could achieve at most would be a deficiency of 11 (vertices) – 1 (connected component) – 6 = 4.



What does all of this actually mean in practical terms? Obviously, we would not expect that the process of "steering communicative discourses" such that they follow the pattern of available stoichiometric compatability classes is one that can be pursued with any conflicts or contradictions showing up in the run of the ongoing procedures. But the important point is that there are *procedures* at all in the first place! It is certainly not possible to eventually achieve a transitory and dynamical equilibrium state without a suitable inventory of procedures. We have to remember that for a usual, not too large, city (we speak of Berlin here), we are talking of about 225 x 10 x 12 = 27.000 groups (species) involved – despite the fact that many individual inhabitants would be multiple group members at the same time. Hence, the concrete presence of various legal procedures (although very often they might appear somewhat unrealistic for the problems in question) defines a useful framework within which communicative processes of discourse can actually take place.
We can recognize that this (tentative) result is compatible with the rule derived from the traditional episode about Heraclitus who when asked for the appropriate means to live together in the polis community did not answer, but put together a barley-drink (the *kykeon* notorious for visitors of the Delphi oracle) instead and drank it down. The idea is in the *stirring*, because the composite parts of the drink separate when the stirring is stopped. In other words, it is necessary to keep everything in motion all the time in order to secure a permanent "stirring" of what composes the "fluid" of society. When looking at the networks above, we can gain some insight into what this "stirring" practically means.
Hence, in the end, when applying aspects of network theory to fields that are far too complex for admitting a closed formal description of their models, such as social systems, we nevertheless extract ethical as well as political results that essentially follow from the structure of the networks involved. It is in this sense that we can talk about the *topology of communication* that is determining the *logic of social space.*